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A Variational Launch Window Study

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In this paper the problem of establishing a circular orbit in a prespecified space-fixed plane is considered with the viewpoint of establishing performance penalties resulting from deviations in launch time from optimum launch time. The purpose is to obtain, as a function of allowable performance loss, a launch window, i.e., a certain time interval about the optimum launch time within which launch may occur, resulting in no more than the allowable performance loss. The analysis is performed using a three-dimensional calculus of variations flight deck for flight out of the atmosphere, resulting in optimum maneuvers both in and out of plane. Hence, the performance figure that results is strictly a function of the launch time. The orbital plane of basic interest is that of minimum inclination obtainable by a two-dimensional injection from Cape Canaveral. Variations of a few degrees below this value are considered to determine the minimal expense for plane changes.

Introduction

THE investigations that are discussed herein were made for a defined vehicle, a Saturn V. It is desired to launch this vehicle from Cape Canaveral into a circular orbit that lies in a specified plane. For this study, the altitude and the plane of the orbit are parameterized, i.e., more than one set of conditions are considered.

Any particular plane for the orbit can be considered as being defined by its inclination to the equatorial plane, for if one considers two planes of the same inclination, any trajectories establishing an orbit in one of the planes will do exactly the same in the other if launch occurs at an appropriately different time of day. The specified plane therefore is definable by a single parameter.

For a reference plane, that inclination for which a circular orbit can be established with the least propellant expenditure is taken. Such a plane would have very nearly the inclination equal in magnitude to the latitude of the launch site, for it would be established with the use of a due-east launch azimuth in order to obtain maximum benefit from the rotation of the earth. But because of the rotation of the earth, it will not be exactly the plane of inclination of the launch site, and the segment of trajectory through the atmosphere will not lie exactly in a plane.

However, these differences are slight in comparison to the variations to be considered. Therefore, bearing in mind that the reference plane actually differs by a slight amount in inclination from the latitude of the launch site, the authors nevertheless will refer to the reference plane as that plane with inclination equal in magnitude to the latitude of the launch site. Furthermore, it will be considered that the trajectory leading to injection into the circular orbit in this plane lies entirely within this plane if launch occurs at precisely that instant of the day when the launch site passes through the reference plane.

Other planes for the circular orbit will be defined by the parameter Δi, which is the difference in inclination from that of the reference plane; Δi is positive for inclinations greater than that of the reference plane.

If the altitude of the circular orbit is denoted by h, it is clear that the desired orbit is specified by the two parameters Δi and h. For this reason, these two parameters will be referred to as the *orbit parameters*. The values of Δi considered are generally less than 2° in magnitude. The values of h considered are 100 km, 100 naut miles, 150 naut miles, and 200 naut miles.

Assume now that a particular set of orbit parameters has been selected. One then faces the problem of determining the best time of day for launch. For example, for the reference plane ($\Delta i = 0$), the best time of day for launch is that instant of the day when the launch site passes through the reference plane, resulting in a two-dimensional trajectory. Variations from this launch time will result in a deterioriation of performance, i.e., in an increase in propellant expenditure. Generally, one can expect that, for each value of Δi under consideration, it is possible to establish a certain best, or optimum, launch time.

Furthermore, it is not reasonable to assume that the launch azimuth is unimportant. Indeed, one of the major results of this study is the demonstration of a significant dependence of performance on launch azimuth. It will be seen that, for each set of orbit parameters, a certain optimum launch azimuth corresponds to each time of launch. Thus, the establishment of the launch window entails the determination of the expense of variations of launch time from the optimum launch time, and, for the final analysis, to each nonoptimum launch time should be associated the optimum launch azimuth for that launch time.

Launch Parameters

It has been seen that both the launch time and the launch azimuth are available for optimization, and the major part of this investigation consists of establishing optimum trajectories

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into the circular orbit under variations of these parameters. These two parameters will be referred to as the *launch parameters* as distinguished from the orbit parameters.

Launch azimuth is in common use and needs no clarification, except that one should agree to measure azimuth clockwise from north. Hence, a due-east launch azimuth is 90°, and due-south is 180°.

On the other hand, a careful and convenient formulation of launch time must be given. This will be done with the aid of Fig. 1.

Neglect the motion of the earth about the sun and consider only the rotation about its axis. Thus, the rotational axis is considered to be a fixed line. Consider a sphere about the center of the earth, which, however, is not rotating. For convenience, the sphere is taken to be the size of the earth, and the equator of the earth is drawn on the sphere.

The locus of the launch site on this sphere as the earth rotates on its axis is, of course, a circle parallel to the equator. This is called the *launch circle*. The plane of the orbit contains the center of the earth. Its projection onto the fixed sphere is, therefore, a great circle.

To consider different launch times, the launch site may be considered to move along the launch circle. If, for the moment, only due-east launch azimuths are considered, the plane of motion of the vehicle at the launch instant will be that plane containing the center of the earth and tangent to the launch circle at the point of launch. At precisely one instant during the day, this plane will have a line of intersection with the equatorial plane identical to that of the desired plane, and the ascending and descending nodes will lie on the same side of the nodal line as do the ascending and descending nodes of the desired orbit. It is logical to refer to the two planes as being *co-nodal* and to call that instant of the day which yields the co-nodal planes the *co-nodal instant*. In Fig. 1, from left to right, are illustrated launch prior to the co-nodal instant, launch at the co-nodal instant, and launch after the co-nodal instant.

Notice that the co-nodal instant corresponds to that point on the launch circle which is midway in longitude between the nodes of the desired orbit. In this way, the co-nodal instant can be characterized without recourse to a due-east launch azimuth. This is important, since it is desirable to consider other launch azimuths. Therefore, the co-nodal instant is defined by its forementioned longitudinal properties.

In the event that launch occurs at the co-nodal instant with a due-east azimuth, it will be called a *co-nodal launch*. Note that the co-nodal launch is the optimum launch for the reference plane.

One is now in a position to define a parameter for launch time. The parameter Δt_L (Δt at launch) is zero at the co-nodal instant, negative for earlier times, and positive for later times.

To summarize, now there are four parameters. The two orbit parameters are the altitude h of the circular orbit and the inclination Δi of the orbit plane relative to the reference plane. The launch parameters are the launch azimuth A_{Z0} and the launch time relative to the co-nodal instant Δt_L. Once values are assigned to these four parameters, the problem that remains is the determination of the optimum trajectory

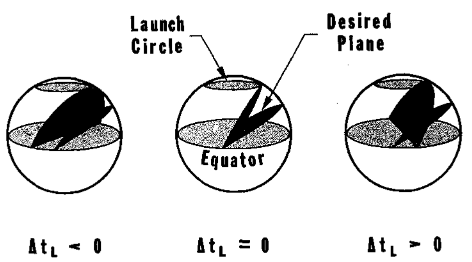

Fig. 1 Geometry resulting from variations in the launch parameter Δt_L

into the orbit. It is in conjunction with this determination that the calculus of variations is discussed, but before beginning the discussion of trajectory shaping, the vehicle itself should be discussed.

Vehicle and Performance

The configuration used in this study is, for the present purposes, a two-stage vehicle. No restart capabilities and no attitude control available for coasts between stages are assumed. Consequently, all trajectories considered will be propelled continuously from launch to injection.

The thrust magnitude in each stage is constant with thrust termination occurring in the first stage at a fixed time and, of course, in the second stage at the appropriate instant for injection.

The performance penalties for the launch window are not expressed directly in terms of weight, but rather in terms of the more general velocity increment. The velocity penalties for each value of h are determined relative to the performance for the co-nodal launch into the reference plane. It is clear that, for each value of h, the largest weight that can be injected into a circular orbit of altitude h is that weight injected into the reference plane with co-nodal launch. For other values of Δi, A_{Z0}, and Δt_L, the difference between this largest weight and the weight injected for the particular values of Δi, A_{Z0}, and Δt_L considered is converted to an equivalent velocity increment called the *velocity penalty* or simply the *velocity increment*. Thus, for each value of h, there is exactly one situation for which the velocity penalty (notation: Δv) is zero.

First Stage

Trajectory shaping in the first stage is not done with the calculus of variations. Instead, a small angle of attack is introduced shortly after ignition in order to initiate tilting. This angle of attack then is removed so that the ensuing trajectory is a so-called "gravity-turn" or "gravity-tilt." In initiating the tilt, the angle of attack is introduced and taken out in a fixed manner and is a certain constant value in between. If this value is denoted by α_0, the entire first-stage shaping is seen in this way to be a function of the single parameter α_0. In conjunction with second stage, the parameter α_0 is optimized later.

Second Stage

Second stages are calculated with a three-dimensional calculus of variations flight deck. The basic equations of this deck are two simultaneous second-order vector differential equations in the position coordinates and thrust components of the vehicle. They are written for a spherical earth and are in Cartesian coordinates. The equations are valid if the coordinate system has its origin at the center of the earth and is space-fixed. Thus, the motion of the center of the earth intrinsically is neglected.

Let $\mathbf{X}$ be the position vector of the vehicle and let $\boldsymbol{\lambda}$ be a vector along the thrust vector. Let r be the length of $\mathbf{X}$, F be the magnitude of the thrust vector, m be the mass of the vehicle, and μ be the Gaussian constant given by $\mu = GM = g_0 r_0^2$.

The first of the two vector equations is the equation of motion:

$$\ddot{\mathbf{X}} = (\boldsymbol{\lambda}/|\boldsymbol{\lambda}|)(F/m) - (\mu/r^3)\mathbf{X}$$

which is valid for any control variable $\boldsymbol{\lambda}(t)$. The second equation is obtained by application of the Euler-Lagrange equation and determines that control history $\boldsymbol{\lambda}(t)$ which is necessary for propellant minimization. For this reason, it is referred to as the control equation:

$$\ddot{\boldsymbol{\lambda}} = -(\mu/r^3)[\boldsymbol{\lambda} - (3\mathbf{X}\cdot\boldsymbol{\lambda}/r^2)\mathbf{X}]$$

The equation of motion and the control equation, considered as a system, define a family of solutions which will be called variational trajectories. A particular variational trajectory is a particular solution to the forementioned system. Each second stage, therefore, is defined by the appropriate particular solution, and the problem now reduces to the numerical determination of that solution.

This determination is, in reality, a two-point boundary value problem. The remaining discussion in this section will be devoted to this problem and its solution.

The family of variational trajectories is generally an 11-parameter family. At first it would seem that the general solution would involve four vector constants or 12 parameters instead of 11, but it should be noted that the two equations are homogeneous in the vector $\boldsymbol{\lambda}$. This means that the initial length of $\boldsymbol{\lambda}$ always might be taken as unity, for example, making $\boldsymbol{\lambda}$ determinable from two of its components (to within the sign of the third component).

In this problem, a set of parameters (h, Δi, A_{Z0}, Δt_L, α_0) is considered to define a first stage and a desired orbit. Since the initial point of the second stage can be considered identical to the end point of the first stage, the initial values of $\mathbf{X}$ and $\dot{\mathbf{X}}$ for the second stage are defined. This then eliminates six of the 11 parameters, leaving a five-parameter family.

But, because the second end point of the second stage is not fixed, but is in fact free to vary over the desired orbit, another condition of the calculus of variations, known as the transversality condition, is applicable. For this problem, the transversality condition comes out to be very simple. If $\mathbf{N}$ denotes a vector normal to the desired plane, the transversality condition states that

$$\mathbf{N}\cdot(\boldsymbol{\lambda} \times \dot{\mathbf{X}} + \mathbf{X} \times \dot{\boldsymbol{\lambda}}) = 0$$

at cutoff. But it can be shown that the quantity $\boldsymbol{\lambda} \times \dot{\mathbf{X}} + \mathbf{X} \times \dot{\boldsymbol{\lambda}}$ is a constant of motion, i.e., it is identically constant on every variational trajectory. Therefore, the transversality condition is equally applicable at the initial point of the second stage. Recall that $\mathbf{X}$ and $\dot{\mathbf{X}}$ are known at the initial point and that $\mathbf{N}$ is determinable from the parameters (h, Δi, A_{Z0}, Δt_L, α_0) (not all are necessary).

All of this means that only four of the six components of the initial values of $\boldsymbol{\lambda}$ and $\dot{\boldsymbol{\lambda}}$ remain to be determined if one wishes to take these as the parameters of the family of solutions, for the homogeneity eliminated one of the six, and, as has just been stated, the transversality will eliminate another.

The conditions to be met at the end point of the second stage—the necessary and sufficient conditions for the specified orbit—now must be discussed. Letting r_0 denote the radius of the earth, the following can be written:

$$\mathbf{X}\cdot\mathbf{X} - (r_0 + h)^2 = 0 \qquad \dot{\mathbf{X}}\cdot\dot{\mathbf{X}} - \frac{\mu}{r_0 + h} = 0$$
$$\mathbf{X}\cdot\dot{\mathbf{X}} = 0 \qquad \mathbf{N}\cdot\mathbf{X} = 0 \qquad \mathbf{N}\cdot\dot{\mathbf{X}} = 0$$

These are five conditions. For each assumed set of values of the four remaining parameters of the family of variational trajectories, one of these five conditions may be used to define the end point of the resulting second stage. It was found to be convenient to cut off each variational trajectory when the second condition (circular velocity) was satisfied.

Then there is a fourth-order isolation problem. There are four independent initial parameters, each of which defines an end point at which four conditions must hold. The development of numerical techniques for carrying out this isolation was a major problem in this analysis. Of those techniques attempted, the most successful was basically a Newton iteration in four variables. The convergence is good; in general, the entire iteration requires less than 20 sec on the 7090. But the region of convergence is generally small and requires a good initial guess. The procedure generally followed was, for this reason, to begin with a known solution and to vary the parameters Δi and Δt_L stepwise in small increments into the set of values for which the solution was desired.

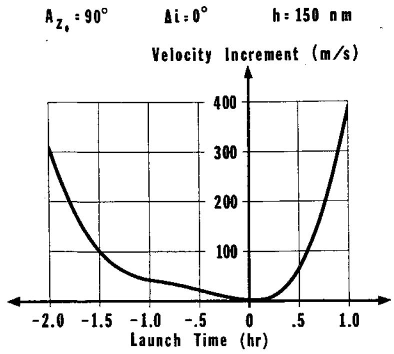

Fig. 2 Velocity increment required for deviations from co-nodal launch

The foregoing now will be summarized. For each set of parameters (h, Δi, A_{Z0}, Δt_L, α_0), one is able to determine (generally) a single particular solution of the family of variational trajectories by means of a fourth-order numerical isolation. From this solution, the velocity penalty for the set (h, Δi, A_{Z0}, Δt_L, α_0) is obtained.

Optimization

One would be led to suspect from physical considerations that the optimization of the parameter α_0 would be dependent primarily on h but not highly sensitive to the parameters Δi, A_{Z0}, and Δt_L for small variations of these parameters. For the variations that have been considered, this is in fact the case; α_0 was optimized for each h for the co-nodal launch into the reference plane. For other considered values of Δi, A_{Z0}, and Δt_L (but for the same value of h), the velocity penalty incurred by not re-optimizing α_0 was less than 1 m/sec.

Therefore, only one value of α_0 was used for each value of h. In this way one is back to the original four parameters: the orbit parameters and the launch parameters.

In presenting the results of the optimization, the authors will consider first that, for every case, $A_{Z0} = 90°$. In this way, some measurement of the importance of optimizing the launch azimuth can be obtained.

Numerical Results

As a simpler problem, consider the problem of establishing the launch window for a 150-naut-mile circular orbit in the reference plane ($\Delta i = 0$) with a due-east azimuth. This window is shown in Fig. 2. The velocity increment is, by definition, zero for the co-nodal launch and is dependent on launch time in the manner shown in the figure. This window is not symmetric and possesses a strange hump in the vicinity of -1 hr. Both the lack of symmetry and the hump will be seen to disappear when the launch azimuth is optimized.

If other inclinations are considered, i.e., other values of Δi, a one-parameter family of curves such as the one shown is obtained. Such a family was generated by considering negative values of Δi to $-1°$. The effect of these variations was to shift the curve up and to the left, gradually take out the hump, and make the curve symmetric, although not about the Δv axis. The optimum launch time moved to negative values. For the case $\Delta i = -1°$, the optimum launch time was about $\frac{1}{2}$ hr prior to co-nodal launch, and the corresponding velocity increment about 65 m/sec.

The same family of curves as is being discussed now was generated for other values of h: $h = 100$ km, 100 naut miles, and 200 naut miles. The qualitative behavior of the curves did not vary, and the numerical differences were relatively small. The largest differences occurred for the largest differences in the values of h. If one compares the families for $h = 100$ km and $h = 200$ naut miles (100 km is roughly 50 naut miles), the following differences are to be seen. For each value of Δi, the optimum launch times differ very little.

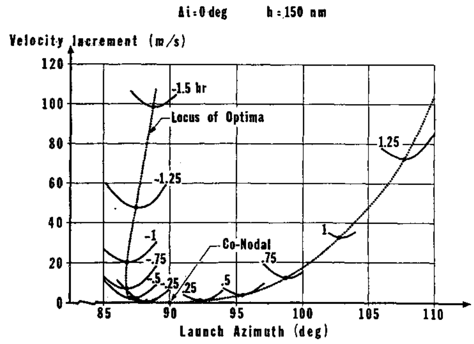

Fig. 3 Velocity increment required for injections into circular orbit of launch latitude vs launch azimuth for different launch times

The biggest difference occurs between the corresponding curves of $\Delta i = -1°$, where the optimum launch times differ by about 4 min. If the curves are compared for values of Δv, again the largest differences are to be found for $\Delta i = -1°$. In considering only that portion of the curves below 100 m/sec, the largest difference is about 15 m/sec. For larger values of Δv, this difference becomes significantly larger. The reason for this is clear from Fig. 2. Because of the steep slope of the upper portion of the curve, a slight displacement of the curve to the left will bring about large changes in the value of Δv.

In performing the study, it was decided to concentrate the analysis on those variations resulting in velocity penalties less than 100 m/sec. With this understanding, it may be stated that the Δv launch window for due-east launches is essentially invariant over the considered values of h.

The next step in the analysis is to investigate the effects of variations in launch azimuth. For a number of reasons, this was done only for the case $h = 150$ naut miles. The main reason was the very large amount of computation required for these investigations. However, the similarity of the launch windows for different altitudes exhibited for the due-east azimuth also was influential in restricting the investigation.

To illustrate the effects of launch azimuth variations, the authors return to the reference plane. In Fig. 3 a plot of velocity increment is shown as a function of launch azimuth for different launch times. Note that for each launch time an optimum launch azimuth is exhibited clearly. The dashed curve shows the locus of the optima.

It already is clear from this figure that much is to be gained in performance through optimization of the launch azimuth, for the velocity increment corresponding to the optimum launch azimuth is the lowest point on each curve, and the velocity increment for a due-east azimuth is given by an extrapolation of each curve to the 90° abscissa. Note that the optimum launch azimuth is due-east for the co-nodal launch, northerly for earlier launches, and southerly for later launches.

The curves in Fig. 3 were generated also for values of Δi from $-0.25°$ to $-1.75°$. As Δi varies from zero, the entire

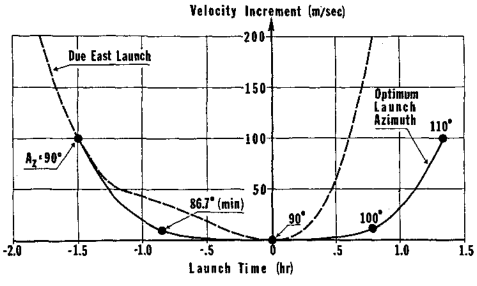

Fig. 4 Velocity increment required for acquisition of space-fixed orbital plane as a function of deviations from optimum launch time on the surface of the earth

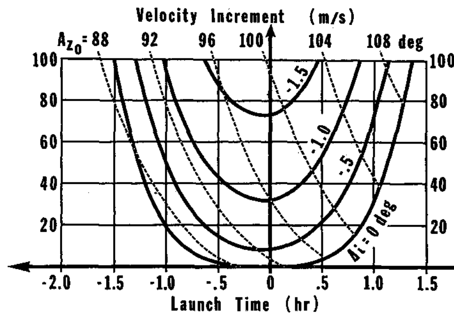

Fig. 5 Velocity increment for 150-naut-mile circle vs launch time for various inclinations and employing optimum launch azimuths

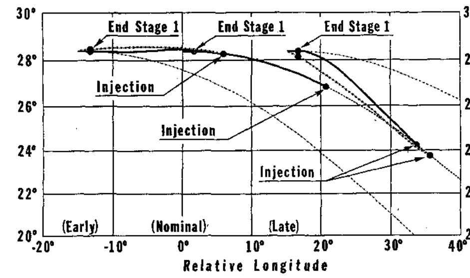

Fig. 6 Three-dimensional second-stage maneuvers required to obtain nominal orbit for early and late launch

curve is raised. The lowest point on the locus of optima would correspond to the velocity increment for optimum launch time with the optimum launch azimuth. This would be the absolutely optimum injection into that plane. For $\Delta i = -1.75°$, the optimum injection required 101-m/sec velocity increment (recall that this is referenced to the co-nodal launch into the reference plane), and for this reason larger variations in Δi were not considered. The optimum figures for different Δi will be seen in a later figure.

With the information in Fig. 3, it is possible to plot the Δv window for the case $\Delta i = 0$. This is shown in Fig. 4, along with the launch window for the due-east launch azimuth for purposes of comparison. As has been pointed out, the window for the optimum azimuth is more nearly symmetric and does not possess the strange hump. It is pleasing to note that the optimized azimuths yield a 1-hr launch window for only 6 m/sec, compared to 25 m/sec for the due-east azimuth. For 2 hr, the corresponding figures are 25 and 85 m/sec.

Figure 5 shows the Δv launch window for optimum launch azimuth and for several values of Δi. This figure embodies the information that the authors initially set out to obtain, especially if it is similar for other orbital altitudes. The dashed curves in this figure are intended to show the behavior of the optimum launch azimuth. For any particular Δi curve, the intersections of that curve with the dashed curve define the optimum launch azimuth for that launch time.

The abscissas of the minimum points of the curves define the optimum launch time as a function of Δi. The corresponding ordinates give the lowest expense for the plane change. The optimum launch time for the highest curve is less than 4 min prior to co-nodal launch. Hence, for the small values of Δi considered and for the optimum launch azimuth, the optimum launch instant remains essentially the co-nodal instant. Recall that, for the due-east launch azimuth, the optimum launch time was $\frac{1}{2}$ hr prior to co-nodal launch for $\Delta i = -1°$.

Figure 6 is intended to give some geometric concept of the involved trajectories. This figure is a Mercator projection of flight paths on a nonrotating earth. In each case, injection is made into a 150-naut-mile circle in the reference plane. There are two trajectories shown for each of three launch times: $\Delta t_L = -1$ hr, $\Delta t_L = 0$, $\Delta t_L = +1$ hr. For each launch time, the solid trajectory is the optimum trajectory with due-east launch azimuth, and the dashed trajectory is for optimized launch azimuth.

The $\Delta t_L = 0$ trajectory is the same for due-east and optimum launch azimuths, since for this launch time the optimum azimuth is due-east. The entire trajectory is seen to lie in the reference plane, and where the trajectory terminates, the reference plane is extended by the dashed curve. For the other two launch times, the reference plane is translated and drawn through the launch site in order to yield a visualization of two-dimensional motion.

Some understanding of the variation of launch azimuth can be obtained from this figure. Consider, for example, the trajectories shown for $\Delta t_L = +1$ hr. For the due-east launch azimuth, the trajectory is "S" shaped. There are two "turnings" or bendings of the trajectory. Near the launch site, the flight plane is turned southerly to bring the motion toward the desired plane. Towards the injection point, a second turning is necessary to bring the velocity vector as well as the position vector into the plane.

Optimization of the launch azimuth is seen to alleviate the necessity for the bending around the launch site. Thus, the bending of the trajectory, i.e., its three-dimensionality, can be equated roughly with energy expenditure.

This would leave one with a final conjecture. The initial bending of the plane seen for the due-east launch azimuth is necessary to direct the motion toward the desired plane, and if this were not done, burnout would occur at some point relatively remote from the reference plane. But if a coast were introduced at a sufficiently high energy level to avoid excessive descent, and if the coast were of a sufficiently long duration to allow a displacement of the order of 180°, the vehicle would approach automatically the reference plane from the other side of the earth. This may improve performance in much the same manner as did optimization of the launch azimuth.

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Investigation of Maximum Stresses in Long, Pressurized, Cylindrical Shells

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Equations are derived for the slope, deflection, and maximum stresses in a cylindrical shell subjected to edge shears and moments, including the simultaneous action of axial loads. A sample problem is given in which the maximum stresses obtained by the refined equations are compared to those obtained by approximate methods currently used in industry in which the axial load restraints on shell rotations and deflections are neglected. A 100-in. radius cylinder with a wall thickness of 0.5 in. and an internal pressure of 1000 psi was selected arbitrarily as a basis for comparison. The maximum hoop stress was found to be 213,770 psi by the standard method and 200,000 psi by the method developed in this paper. The reduction in maximum hoop stress is due to the restraining influence of the axial loads on hoop displacements of the shell. Where warranted, the use of the refined method generally will give lower calculated maximum stresses and result in a lighter-weight shell design.

Introduction

A refined analysis is presented which accounts for the axial load effect on the maximum stresses in a long, pressurized, cylindrical shell. A comparison between the refined analysis and the standard analysis currently used in industry is given. This paper reports the first phase of an investigation that ultimately will establish parameters defining the relative merits of each analysis.

Discussion

Figure 1 shows a cylindrical shell constituting the central portion of a pressure vessel subjected to internal pressure p and three edge loads at the left end: moment M, shear V, and axial load N. Values of M and V are obtained by performing a compatibility analysis between the cylindrical shell and the adjoining structure. Wei[1] has presented the theory and formulas used in establishing the compatibility equations and shows their application in stress-analyzing solid propellant rocket cases. In addition, formulas for obtaining the maximum meridional (axial) and hoop (circumferential) stresses in the cylindrical shell (Fig. 1) were derived by Wei. These formulas are repeated in Tables 1 and 2 for convenience.

Hetenyi[2] points out that current stress analysis practice is to use formulas that neglect the effect of the axial load N on shell rotations and deflections due to edge loads M and V. Maximum stresses calculated using these standard formulas in the compatibility equations compare reasonably well with experimental stresses for rocket case (or pressure vessel) sizes currently used in industry.[1] It is conceivable, however, that, in the near future, vessel sizes will be of such diameter and wall thickness that a more refined analysis, i.e., considering the effect of the axial load on shell slope, deflection, and stress, must be used.

A set of formulas has been derived which accounts for the axial load effect and which gives 1) the slope and deflection at the edge of a cylindrical shell due to edge moment and shear loads, 2) the location of points of zero slope on the meridional and hoop stress curves, and 3) the magnitude of the meridional and hoop stresses. The formulas are based on the similarity between a cylindrical shell and a beam on

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